number of possibilities *for:* *picking 'r' out of 'n' items* *or:* *'n' items taken 'r' at a time*	***repeats allowed*** (i.e., we can pick the same item over and over again)	***no repeats allowed*** (i.e., once an item is picked, it cannot be picked again)
permutations (i.e., order matters)	n^r	$P(n,r) = {}_nP_r = {}^nP_r = \frac{n!}{(n-r)!}$
combinations (i.e., order does not matter)	$\binom{r+n-1}{r} = \frac{(r+n-1)!}{r!\,(n-1)!}$	$\binom{n}{r} = C(n,r) = {}_nC_r = {}^nC_r = \frac{n!}{r!\,(n-r!)}$

When there are duplicates (n1×m1, n2×m2, n3×m3, ...) in the 'n' items, the number of possibilities are reduced by:

$$\frac{original\ possibilities}{m_1! \cdot m_2! \cdot m_3! \cdot \ldots}$$

Helpful Identities:

$$0! = 1! = 1$$
$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

Example: 7! = 7 · 6 · 5 · 4 · 3 · 2 · 1 = 5040

$$P(n,n) = n!$$
$$C(n,r) = C(n,n-r)$$
$$C(r+n-1,r) = C(r+n-1,n-1)$$

probabilities

If P(E) represents the probability of an event E occurring, then, we have,

P(E) = 0 if and only if E is an impossible event
P(E) = 1 if and only if E is a certain event
0 ≤ P(E) ≤ 1

Sample space(S) is the set of all of the possible outcomes of an experiment; n(S) represents the total number of (equal probability) outcomes in the sample space; n(E) represents the number of outcomes favorable to event E.

P(E) = n(E) / n(S) ==> probability that event E will occur
P(E') = 1 - (n(E) / n(S)) ==> probability that event E will not occur.
Therefore, now we can also conclude that, P(E) + P(E') = 1

Set notation:

"∪" ==> or/union (this looks like the letter "U")
"∩" ==> and/intersection (this looks like an up-side-down letter "U")

"addition rule of probability" -- probability of A or B occurring:

P(A or B) = P(A ∪ B) = P(A) + P(B) - P(A ∩ B)

Conditional Probability Formula -- "probability of A given that B has already occurred":

P(A | B) = P(A / B) = $P_B(A)$ = P(A ∩ B) / P(B) ==> note that P(B) ≠ 0, since B has already occurred

"multiplication rule of probability" -- probability of A and B occurring:

P(A and B) = P(A ∩ B) = P(A) · P(B) ==> in case of independent events
P(A and B) = P(A ∩ B) = P(A) · P(B | A) ==> in case of dependent events

Therefore:

P(A occurring 3 times in a row) = P(A) · P(A) · P(A) = $(P(A))^3$
P(A occurring N times in a row) = $(P(A))^N$

example 1

We flip a coin 3 times... Repeats are obviously allowed and we assume order does not matter... It is like picking an item out of a bag containing 2 items (i.e., n = 2), one item representing heads and the other item representing tails. We put the item back in the bag after we pick it so it can possibly be picked over and over again. We pick out of the bag 3 times (i.e., r = 3). Using the formula above (order does not matter, repeats allowed):

$$\frac{(r+n-1)!}{r!\,(n-1)!} = \frac{(3+2-1)!}{3!\,(1)!} = \frac{4!}{3!} = \frac{24}{6} = 4$$

So we find there are 4 possible outcomes, but these 4 outcomes do not have equal probability (so we cannot yet use the probability formulas above). By examination, these outcomes are:

HHH (i.e., 3 heads), HHT (i.e., 2 heads and a tail), TTH (i.e., two tails and a head), TTT (i.e., 3 tails)

If order does matter, on the other hand, we would use the other formula above (order matters, repeats allowed):

$$n^r = 2^3 = 8$$

And we find there are now 8 equally probable outcomes, which by examination are:

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

So now we can use:

P(E) = n(E)/n(S)

Resulting in:

P(HHH) = 1/8 = 12.5% chance
P(HHT) = 3/8 = 37.5% chance (since there are 3 ways we can get 2 heads and a tail)
P(TTH) = 3/8 = 37.5% chance (since there are 3 ways we can get 2 tails and a head)
P(TTT) = 1/8 = 12.5% chance total = 8/8 = 100% chance

example 2

We have 20 students in a class and a bus that holds only 16 students, so 4 students will have to drive in a separate car. How many total ways can we select 4 students (i.e., r = 4) out of 20 (i.e., n = 20) to fill the car? Repeats are not allowed (since each student can only be picked once!) and order does not matter (yet). Using the combination formula above:

$$\frac{n!}{r!\,(n-r!)} = \frac{20!}{4!\,(20-4)!} = \frac{20!}{4!\,16!} = \frac{20\cdot 19\cdot 18\cdot 17}{4\cdot 3\cdot 2\cdot 1} = \frac{116280}{24} = 4845$$

How many total ways can we select 16 students (i.e., r = 16) out of 20 (i.e., n = 20) to fill the bus? Using the same formula:

$$\frac{n!}{r!\,(n-r!)} = \frac{20!}{16!\,(20-16)!} = \frac{20!}{16!\,4!} = \frac{20\cdot 19\cdot 18\cdot 17}{4\cdot 3\cdot 2\cdot 1} = \frac{116280}{24} = 4845$$

Notice the same number! We can also find these using the combination function on our calculator:

$$C(20,4) = {}_{20}C_4 = {}^{20}C_4 = 4845 \qquad C(20,16) = {}_{20}C_{16} = {}^{20}C_{16} = 4845$$

What is the probability that George, Mary, Sally, and Tim are randomly selected to be in the car together?

P(GMST) = n(GMST)/n(Total) = 1/4845 = 0.02%

Assuming they are in the car, how many total ways can George, Mary, Sally, and Tim (n = 4) be arranged in the 4 seats of the car (r = 4)? Now order matters, so we use the permutation formula and helpful identity above:

$$P(4,4) = {}_4P_4 = {}^4P_4 = 4! = 4\cdot 3\cdot 2\cdot 1 = 24$$

What are the odds that Mary gets the front seat?

P(Mfront) = n(Mfront)/n(Total) = 3!/4! = 6/24 = 0.25 = 25% (you probably could have guessed this!)